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A FINITE-DIFFERENCE SOLUTION TO A MIXED-TYPE PARTIAL DIFFERENTIAL EQUATION: AN OCEAN DYNAMIC MOTION MODEL

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Abstract—A mixed-type (hyperbolic, parabolic, elliptic) partial differential equation, which results from a model of a towed cylindrical acoustic antenna in the ocean, has been solved by combining the techniques of the method of lines and the numerical ordinary differential equation method. The model describes the transverse dynamic motion of a zero bending rigidity cylinder in an axial viscous stream. This initially ill-posed problem was satisfactorily solved bv the above approach. The efficiency of this numerical solution has been improved over previous solutions. Discussions of these improvements are the main theme of this paper which presents an explicit finite-difference scheme; the improvements increase computation speed while maintaining the required accuracy. The improvements are discussed both theoretically and computationally with respect to consistency, stability and convergence. The results of a numerical test example are given. in this report (KR)

1. INTRODUCTION

A partial differential equation (PDE) describing the transverse dynamics of a long cylinder in axial flow was first obtained by Paidoussis [1]. This dynamic motion equation is a fourth-order PDE. In many problems of interest the aspect (length-to-diameter) ratio is quite large and the excitation forces are much larger than the diameter. In these cases the flexural stiffness of the cable is negligible relative to the tension stiffness term. Neglecting this term results in a mainstream equation which is a mixed type; varying from a hyperbolic equation to a parabolic equation elliptic equation. Over a decade ago, a solution to this problem was proposed by Ortloff and Ives [2] using the method of separation of variables. Their solution contained Bessel functions whose order and argument are complex and of a magnitude not well approximated by asymptotic solutions. Moreover, the treatment of the free-end boundary condition was through a "bounded" criteria which is not applicable to numerical solutions. The Ortloff and Ives solution [2] is also quite inefficient for arbitrary excitation time services because the solution must be repeated for each frequency present in the Fourier services of the excitation term. Recently, Lee and Kennedy [3] made the problem numerically well-posed by introducing an appropriate treatment to the free-end boundary condition. Then, a combination of the method of lines and a numerical ordinary differential equation (ODE) method (generalized Adams methods) were applied to solve the well-posed problem. Details of the theoretical and numerical developments can be found in Ref. [3]. The efficiency of the above solution can be further improved by a finite-difference scheme. The main theme of this paper is an improved solution using a basic explicit finite-difference scheme. Further, this development is proved theoretically convergent. Prior to the convergence proof, the theory with regard to stability is fully discussed. The consistency condition is examined. The validity of the finite-difference solution is demonstrated by an exact solution test which is used to examine the accuracy of the mathematical formulation as well as computational procedures.

2. PROBLEM AND SOLUTION SUMMARY

A fourth-order PDE [1] representing the dynamic motion of a cylinder in a viscous axial flow form is as follows:

$$EI\frac{\partial^4 y}{\partial x^4} + (M + m)\frac{\partial^2 y}{\partial t^2} + MU^2\frac{\partial^2 y}{\partial x^2} + 2MU\frac{\partial^2 y}{\partial t\partial x} - \frac{\partial}{\partial x}\left[\frac{c_T}{2}\left(\frac{M}{D}\right)U^2(L - x)\frac{\partial y}{\partial x}\right]$$
$$+ \frac{1}{2}c_N\frac{M}{D}U\left(\frac{\partial y}{\partial t}\right) + U\left(\frac{\partial y}{\partial x}\right) = 0. \quad (1)$$

The parameters appearing in equation (1) have the following definitions:

EI = bending rigidity.
M = the effective mass of fluid "pushed" by the cylinder per unit length of cylinder,
m = mass of the cylinder per unit length,
U = fluid velocity,
c_T = longitudinal drag coefficient due to hydrodynamic forces acting along the cylinder,
L = total length of the cylinder

and

c_N = normal drag coefficient due to hydrodynamic forces acting normal to the cylinder.

After neglecting the flexural rigidity and rearranging terms, equation (1) is reduced to

$$\frac{\partial^2 y}{\partial x^2}\frac{M + m}{M} + \frac{\partial^2 y}{\partial x^2}\left[U^2 - \frac{1}{2}c_T\frac{U^2}{D}(L - x)\right] + \frac{\partial y}{\partial x}\frac{U^2}{2D}(c_T + c_N) + 2U\frac{\partial^2 y}{\partial x\,\partial t} + \frac{1}{2}c_N\frac{\partial y}{\partial t} = 0. \quad (2)$$

Ortloff and Ives [2] used the non-dimensional terms

$$\tau = (t/L)U, \quad \beta = M/(M + m), \quad \xi = x/L, \quad \varepsilon = L/D \quad \text{and} \quad \eta = y/L.$$

Substitution of these terms into equation (2) yields

$$\frac{\partial^2 \eta}{\partial \tau^2} + \beta\left[1 - \frac{1}{2}c_T\varepsilon(1 - \xi)\right]\frac{\partial^2 \eta}{\partial \xi^2} + 2\beta\frac{\partial^2 \eta}{\partial \xi\,\partial \tau} + \frac{(c_T + c_N)}{2}\varepsilon\beta\frac{\partial \eta}{\partial \xi} + \frac{1}{2}c_N\varepsilon\beta\frac{\partial \eta}{\partial \tau} = 0. \quad (3a)$$

The dynamic motion equation problem is described by equation (3a). The associated initial and boundary conditions are:

$$\eta(0, \xi) = \eta_1(\xi), \quad \text{describing initial deflection;} \quad (3b)$$

$$\left.\frac{\partial \eta}{\partial \tau}\right|_{\tau = 0} = 0, \quad \text{zero initial velocity,} \quad (3c)$$

$$\eta(\tau, 0) = 0, \quad \text{fixed-end condition;} \quad (3d)$$

$$|\eta(\tau, 1)| < \infty, \quad \text{bounded free-end deflection.} \quad (3e)$$

The set of equations (3a–e) poses an initial-boundary problem in the region of interest (Fig. 1).

Using the method of characteristics we may classify the solution types by evaluating the characteristic type. To do this we evaluate the quadratic equation

$$\beta^2 - \beta\left[1 - \frac{1}{2}c_T\varepsilon(1 - \xi)\right] \begin{array}{lll} > 0 & \xi < \xi^* & \text{hyperbolic} \\ = 0 & \xi = \xi^* & \text{parabolic} \\ < 0 & \xi > \xi^* & \text{elliptic,} \end{array}$$

for some $0 \leqslant \xi^* < 1$. The point to be noted is that over the length of the physical cylinders all three characteristic types are found. The above summarizes the problem background, the region

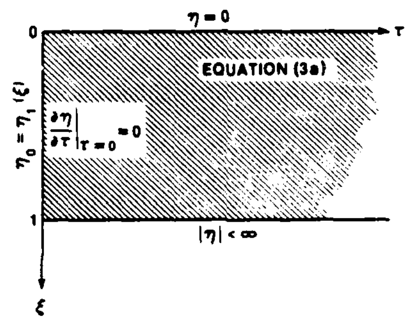

Fig. 1. Region of interest.

of interest and the change of equation type.

Two solutions exist for equation (3a). One is the Bessel function solution, obtained by means of the separation of variables [2]. This solution is inefficient for arbitrary temporal excitation of the cylinder structure. Lee and Kennedy [3] introduced the numerical solution to remove the above inefficiency. Lee and Kennedy introduced the change of variables

$$t = \tau \tag{4}$$

and

$$x = x(\xi, \tau) = \beta\tau + (1 - \xi), \tag{5}$$

which brings equation (3a) into an equivalent form:

$$\hat{\eta}_{tt} + \hat{a}(x,t)\hat{\eta}_{xx} + \hat{b}\eta_x + \hat{c}\hat{\eta}_t = 0, \tag{6a}$$

where

$$\hat{a}(x,t) = \beta\left[1 - \frac{1}{2}c_T\varepsilon(x - \beta t)\right] - \beta^2,$$

$$\hat{b}(x,t) = \frac{1}{2}\varepsilon\beta[c_N\beta - (c_T + c_N)],$$

$$\hat{c} = \frac{1}{2}c_N\varepsilon\beta$$

and

$$\hat{\eta}(x,t) = \eta(\beta\tau + (1 - \xi), \tau).$$

The associated initial and boundary conditions are:

$$\hat{\eta}(x,0) = \eta_1(1 - x), \quad \text{prescribed initial deflection;} \tag{6b}$$

$$\hat{\eta}_t(x,t)|_{t=0} = \beta\frac{\partial}{\partial x}\eta_1(1 - x), \quad \text{initial velocity;} \tag{6c}$$

$$\hat{\eta}(x,t) = f(t) \text{ at } x = \beta t + 1, \quad \text{prescribed fixed-end condition;} \tag{6d}$$

$$|\hat{\eta}(x,t)| < \infty \text{ at } x = \beta t, \quad \text{bounded free-end condition.} \tag{6e}$$

The set of equations (6a–e) defines an initial-boundary problem counterpart to equation (3a).

The ODE solution to equation (6a), proposed by Lee and Kennedy [3], uses the method of lines to discretize partial derivatives with respect to x by central and forward differences. This discretization formulates the PDE, equation (6a), into a system of second-order ODEs in the variable t, i.e.

$$(\hat{\eta}_{tt})_m + \hat{c}(\hat{\eta}_t)_m + \left[\frac{-2\hat{a}(x_m,t)}{h^2} - \frac{c_T\varepsilon\beta + \hat{b}}{h}\right]\hat{\eta}_m + \frac{\hat{a}(x_{m+1},t)}{h^2}\hat{\eta}_{m+1} + \left[\frac{a(x_{m-1},t)}{h^2} + \frac{c_T\varepsilon\beta + \hat{b}}{h}\right]\hat{\eta}_{m-1} = 0, \tag{7}$$

where $h = \Delta x$.

Then, another change of variable, $\frac{\partial\hat{\eta}_m}{\partial t} = W_m$, brings equation (7) into a system of first-order ODEs in the form

$$\hat{\eta}' = \mathbb{A}(x,t)\hat{\eta} + \mathbf{g}(x,t,\hat{\eta}), \tag{8}$$

where the vector $\mathbf{g}$ contains two non-zero components which supply the fixed-end and free-end boundary conditions.

Because the matrix $\mathbb{A}(x,t)$ turns out to be independent of t, the numerical solution to equation (8) with associated conditions can be obtained explicitly by means of the generalized Adams method [3]:

$$\hat{\eta}_{n+1} = e^{\mathbb{A}h}\hat{\eta}^n + h(\mathbb{A}h)^{-1}(e^{\mathbb{A}h} - \mathbb{I})\mathbf{g}_n, \tag{9}$$

which summarizes the numerical solution to problem (3a).

Note that the computation of equation (9) is straightforward provided that the matrix exponential $e^{\mathbb{A}h}$ can be carried out simply. An accurate and economical computation of $e^{\mathbb{A}h}$ is by means of a second-order diagonal Pade approximation:

$$e^{\mathbb{A}h}\left[\mathbb{I} + \frac{1}{2}\mathbb{A}h + \frac{1}{12}(\mathbb{A}h)^2\right]^{-1}\left[\mathbb{I} - \frac{1}{2}\mathbb{A}h + \frac{1}{12}(\mathbb{A}h)^2\right]. \tag{10}$$

Even though the $e^{\mathbb{A}h}$ need be calculated only once, it is necessary at every time step to multiply the matrix by a vector. For a matrix of order M, it needs at least $2M^3$ operations initially to obtain the approximation for the $e^{\mathbb{A}h}$, then at every time step there is another M^2 operations needed to produce the result. In view of the cost of the computation, in both speed and computer storage, we propose an explicit finite-difference solution to problem (6a). We discuss the complete finite-difference development in the following sections.

3. A STABLE FINITE-DIFFERENCE MODEL

In this section we describe the complete development of a finite-difference scheme for the solution of problem (6a). It begins with the formulation. Following the formulation, the theory with regard to consistency, stability and convergence is presented for the proposed finite-difference scheme.

3.1. *An Explicit Finite-difference Formulation*

To maintain generality, let us consider the non-homogeneous equation

$$\hat{\eta}_{tt} + \hat{a}\hat{\eta}_{xx} + \hat{b}\hat{\eta}_x + \hat{c}\hat{\eta}_t = F(x,t). \tag{11}$$

where $F(x, t)$ physically models a FORCING function. Pursuing the finite-difference discretization of equation (11), let us derive a partial differentiation expression for $\hat{a}\hat{\eta}_{xx}$:

$$(\hat{a}\hat{\eta})_{xx} = (\hat{a}_x\hat{\eta} + \hat{a}\hat{\eta}_x)x$$
$$= \hat{a}_{xx}\hat{\eta} + 2\hat{a}_x\hat{\eta}_x + \hat{a}\hat{\eta}_{xx}. \tag{12}$$

Since

$$\hat{a} = \beta\left[1 - \frac{1}{2}c_T\varepsilon(1 + x - \beta t)\right] - \beta^2,$$

then

$$\hat{a}_x = -\frac{1}{2}c_T\varepsilon\beta$$

and

$$\hat{a}_{xx} = 0,$$

therefore equation (12) becomes

$$(\hat{a}\hat{\eta})_{xx} = \hat{a}\hat{\eta}_{xx} - c_T\varepsilon\beta\hat{\eta}_x,$$

which implies that

$$\hat{a}\hat{\eta}_{xx} = (\hat{a}\hat{\eta})_{xx} + c_T\varepsilon\beta\hat{\eta}_x. \tag{13}$$

Now, apply the conventional finite-difference discretization to equation (11), making use of equation (13), using the central difference for the second derivative and the backward difference for the first derivative, we obtain a different equation for equation (11), viz.

$$\frac{\hat{\eta}_m^{n+1} - 2\hat{\eta}_m^n + \hat{\eta}_m^{n-1}}{k^2} + \frac{\hat{a}_{m+1}^n\hat{\eta}_{m+1}^n - 2\hat{a}_m^n\hat{\eta}_m^n + \hat{a}_{m-1}^n\hat{\eta}_{m-1}^n}{h^2}$$
$$- c_T\varepsilon\beta\frac{\hat{\eta}_m^n - \hat{\eta}_{m-1}^n}{h} + \hat{b}\frac{\hat{\eta}_m^n - \hat{\eta}_{m-1}^n}{h} + \hat{c}\frac{\hat{\eta}_m^{n+1} - \hat{\eta}_m^n}{k} = F(x_m, t^n), \tag{14}$$

where $k = \Delta t$, $h = \Delta x$.

A rearrangement of equation (14) will result in the simple expression

$$\hat{\eta}^{n+1} + P\hat{\eta}_m^n + Q\hat{\eta}_{m-1}^n + R\hat{\eta}_{m+1}^n + \hat{\eta}_m^{n-1} = SF(x_m, t^n), \tag{15}$$

where

$$P = \left[-2 - 2\hat{a}_m^n\left(\frac{k}{h}\right)^2 - c_T\varepsilon\beta\frac{k^2}{h} - \hat{b}\frac{k^2}{h} - k\hat{c}\right](1 + k\hat{c}), \tag{16}$$

$$Q = \left[\hat{a}_{m-1}^n\left(\frac{k}{h}\right)^2 + c_T\varepsilon\beta\frac{k^2}{h} + \hat{b}\frac{k^2}{h}\right](1 + k\hat{c}), \tag{17}$$

$$R = \left[\left(\frac{k}{h}\right)^2\hat{a}_{m+1}^n\right](1 + k\hat{c}) \tag{18}$$

and

$$S = k^2/(1 + k\hat{c}). \tag{19}$$

Then

$$\hat{\eta}_m^{n+1} = SF(x_m, t^n) - P\hat{\eta}_m^n - Q\hat{\eta}_{m-1}^n - R\hat{\eta}_{m+1}^n - \hat{\eta}_m^{n-1}. \tag{20a}$$

Equation (20a) is the explicit finite-difference scheme for the solution of equation (11). Initially, to compute the η_m^{n+1}, the associated initial and boundary conditions (6b–e) must be fulfilled:

$$\hat{\eta}_m^{n-1} = \eta_1(1 - x_m), \quad \text{the prescribed initial deflection;} \tag{20b}$$

$\hat{\eta}_m^n$ is obtained making use of the initial velocity information, such that

$$\hat{\eta}_m^n = \hat{\eta}_m^{n-1} + \Delta t\beta \frac{\partial}{\partial x}\eta_1(1 - x)\bigg|_m; \tag{20c}$$

$$\hat{\eta}_{m-1}^n = f(t) \text{ at } x = \beta t + 1, \quad \text{the prescribed fixed-end condition;} \tag{20d}$$

$$\left|\hat{\eta}_{m+1}^n\right| < \infty \text{ at } x_{m+1} = \beta t, \quad \text{the bounded free-end condition.} \tag{20e}$$

Problem (20a) with associated conditions (20b–e) constitutes a counterpart initial-boundary problem to problem (6a).

3.2. Consistency

In examining whether the difference equation (15) is consistent with the differential equation (11), we expand $\hat{\eta}_m^{n+1}$ and $\hat{\eta}_m^{n+1}$ in powers of k upon $\hat{\eta}_m^n$, and expand $\hat{\eta}_{m+1}^n$, $\hat{\eta}_{m-1}^n$ in powers of h upon $\hat{\eta}_m^n$.

For rotational economy we drop the indices n and m so that $\hat{\eta} = \hat{\eta}_m^n$. Then

$$\hat{\eta}_m^{n+1} = \hat{\eta} + k\hat{\eta}_t + \frac{k^2}{2!}\hat{\eta}_{tt} + \cdots, \tag{21}$$

$$\hat{\eta}_m^{n-1} = \hat{\eta} - k\hat{\eta}_t + \frac{k^2}{2!}\hat{\eta}_{tt} - \cdots, \tag{22}$$

$$\hat{\eta}_{m+1}^n = \hat{\eta} + h\hat{\eta}_x + \frac{h^2}{2!}\hat{\eta}_{xx} + \cdots \tag{23}$$

and

$$\hat{\eta}_{m-1}^n = \hat{\eta} - h\hat{\eta}_x + \frac{h^2}{2!}\hat{\eta}_{xx} + \cdots. \tag{24}$$

To examine the consistency, we localize the coefficients $\hat{a}_j$ in the closed interval $[t, t + \Delta t]$ using $\bar{a}$ to represent the average $\hat{a}_j$. We rearrange equation (14) and put all terms on the l.h.s. of the equation to give

$$\begin{aligned} &h^2(\hat{\eta}_m^{n+1} - 2\hat{\eta}_m^n + \hat{\eta}_m^{n-1}) + k^2\bar{a}(\hat{\eta}_{m+1}^n - 2\hat{\eta}_m^n + \hat{\eta}_{m-1}^n) \\ &\quad - (c_T\varepsilon\beta - \hat{b})(\hat{\eta}_m^n - \hat{\eta}_{m-1}^n)k^2h + \hat{c}kh^2(\hat{\eta}_m^{n+1} - \hat{\eta}_m^n) - k^2h^2F(x_m, t) = 0. \end{aligned} \tag{25}$$

Substituting equations (21)–(24) into equation (25), we obtain

$$k^2h^2\hat{\eta}_{tt} + k^2h^2\bar{a}\hat{\eta}_{xx} - (c_T\varepsilon\beta - \hat{b})k^2h^2\hat{\eta}_x + \frac{1}{12}kh^2\hat{\eta}_{tttt} + k^2h^4\bar{a}\hat{\eta}_{xxxx} + (c_T\varepsilon\beta - \hat{b})k^2h^3\hat{\eta}_x + \hat{c}k^2h^2\hat{\eta}_y + \frac{1}{2}k^3h^2\hat{\eta}_{tt} - k^2h^2F(x_m, t) = 0.$$

Then

$$k^2h^2[\hat{\eta}_{tt} + (\bar{a}\hat{\eta}_{xx} - c_T\varepsilon\beta\hat{\eta}_x) + \hat{b}\hat{\eta}_x + \hat{c}\hat{\eta}_t - F(x_m, t)] = -\frac{\hat{c}}{2}k^3h^2\hat{\eta}_{tt} - (c_T\varepsilon\beta - \hat{b})k^2h^3\hat{\eta}_{xx} + \frac{k^4}{12}h^2\hat{\eta}_{ttt} - \frac{k^2h^4}{12}a\hat{\eta}_{xxx}; \tag{26}$$

the truncation error E is given by

$$E = -\frac{\hat{c}}{2}k\hat{\eta}_{tt} - (c_T\varepsilon\beta - \hat{b})h\hat{\eta}_{xx} + \frac{k^2}{12}\hat{\eta}_{tttt} - \frac{h^2}{12}\bar{a}\hat{\eta}_{xxxx} + \text{high-order terms} = O(k, h).$$

In the $\lim_{h,k \to 0} E = 0$, which establishes the consistency.

3.3. Stability

From equation (14), if one compares the terms, in equation (14), between $O\left(\frac{1}{k^2}\right)$ and $O\left(\frac{1}{k}\right)$, it is easily seen that as $k \to 0$, the dominant term is $O\left(\frac{1}{k^2}\right)$. Similarly, the dominant term between $O\left(\frac{1}{h^2}\right)$ and $O\left(\frac{1}{h}\right)$ belongs to the $O\left(\frac{1}{h^2}\right)$ term. This analysis suggests that for the stability analysis, one can use

$$\hat{\eta}_{tt} + \hat{a}(x)\hat{\eta}_{xx} = 0 \tag{27}$$

to derive the sufficient condition of stability.

Theorem

The finite-difference scheme (14) used to solve the dynamic motion equation (11) is stable if

$$\frac{k}{h} \leqslant \frac{1}{|\bar{a}|}.$$

Proof. For brevity, write $\lambda = e^{\alpha k}$. Since $\hat{a}(x)$ is not rapidly varying in the variable x, we adopt the "frozen variable" approximation to localize $\hat{a}(x)$ as a constant in x between the interval $[t, t + \Delta t]$ and also for brevity, write $\hat{a}(x) = \bar{a} = \text{const}$. We look for the solution $\hat{\eta}(x, t)$ in the form $e^{\alpha t}e^{i\omega x}$, using the Von Neumann approach. Substituting $\hat{\eta} = e^{\alpha t}e^{i\omega x}$ into equation (27), using the approximation $\bar{a} = \hat{a}(x)$, one finds

$$\frac{e^{\alpha(n+1)} - 2e^{\alpha nk} + e^{\alpha(n-1)k}}{k^2} \cdot e^{i\omega mh} + \bar{a}\frac{e^{i\omega(m+1)h} - 2e^{i\omega mh} + e^{i\omega(m-1)h}}{h^2} \cdot e^{\alpha nk} = 0.$$

A simplification of the above equation gives

$$\lambda^2 + K\lambda + 1 = 0, \tag{28}$$

where

$$K = 2\left\{\bar{a}\frac{k^2}{h^2}[\cos(\omega h) - 1] - 1\right\}.$$

Then

$$\lambda = \frac{-K \pm \sqrt{K^2 - 4}}{2}. \tag{29}$$

The stability condition requires that $|\lambda| \leqslant 1$, which implies that both of the following conditions must be satisfied:

$$|\lambda_1| = \left|\frac{-K + \sqrt{K^2 - 4}}{2}\right| \leqslant 1 \tag{30}$$

and

$$|\lambda_2| = \left|\frac{-K - \sqrt{K^2 - 4}}{2}\right| \leqslant 1. \tag{31}$$

Both conditions (30) and (31) have the term $\sqrt{K^2 - 4}$ in common. This is the term that needs to be analyzed in order to derive the stability condition. Condition (30) is equivalent to

$$-1 \leqslant -\frac{K}{2} + \frac{1}{2}\sqrt{K^2 - 4} \leqslant 1. \tag{32}$$

There are two cases to examine:

$$\text{(i)} \quad K^2 - 4 > 0 \tag{33}$$

and

$$\text{(ii)} \quad K^2 - 4 \leqslant 0. \tag{34}$$

Case (i):

$$K^2 - 4 > 0 \rightarrow K^2 > 4 \rightarrow \left[2\left\{\bar{a}\frac{k^2}{h^2}[\cos(\omega h) - 1]\right\}\right]^2 > 4.$$

In order that the above condition is satisfied, it is necessary that $[\cos(\omega h) - 1] \neq 0$. Since $\bar{a} < 0$ in the hyperbolic region and $[\cos(\omega h) - 1] < 0$, together they require that

$$\bar{a}\frac{k^2}{h^2}[\cos(\omega h) - 1] > 1,$$

which implies that

$$\frac{k^2}{h^2} > \{\bar{a}[\cos(\omega h) - 1]\}^{-1}. \tag{35}$$

Returning to condition (32), we examine

$$-\frac{K}{2} + \frac{1}{2}\sqrt{K^2 - 4} \leqslant 1 \rightarrow K \geqslant -8, \tag{36}$$

which always exists.

Also,

$$-1 \leqslant -\frac{K}{2} + \frac{1}{2}\sqrt{K^2 - 4} \rightarrow K \geqslant 2, \tag{37}$$

since condition (36) is always satisfied; which also implies that $-\frac{K}{2} - \frac{1}{2}\sqrt{K^2 - 4} \leqslant 1$ is obviously true.

Next,

$$-1 \leqslant -\frac{K}{2} - \frac{1}{2}\sqrt{K^2 - 4} \rightarrow K \leqslant 2. \tag{38}$$

On close examination of conditions (35)–(38), it is clearly seen that the case $K^2 - 4 > 0$ will not lead to a condition of stability.

Case (ii):

$$K^2 - 4 \leqslant 0 \rightarrow \left| -\frac{K}{2} + \frac{1}{2}\sqrt{K^2 - 4} \right| \leqslant 1$$

$$\rightarrow \left(-\frac{K}{2}\right)^2 + \left(\frac{1}{2}\sqrt{4 - K^2}\right) \leqslant 1$$

$$\rightarrow \left| -\frac{K}{2} + \frac{1}{2}\sqrt{K^2 - 4} \right| = 1 \quad \text{identically.}$$

Therefore, $K^2 \leqslant 4$ gives

$$\bar{a}\frac{k^2}{h^2}[\cos(\omega h) - 1] \leqslant 2 \rightarrow \frac{k^2}{h^2} \leqslant 2(\bar{a})^{-1}[\cos(\omega h) - 1]^{-1}, \tag{39}$$

$\max[\cos(\omega h) - 1] = 2$ gives [equation (39)]

$$\frac{k^2}{h^2} \leqslant \frac{1}{\bar{a}} \rightarrow \frac{k}{h} \leqslant \frac{1}{\sqrt{\bar{a}}}, \tag{40}$$

establishing the condition of stability. Q.E.D.

3.4. Convergence

The consistency condition, established in Section 3.2, states that the finite-difference solution approaches the true solutions as $k, h \rightarrow 0$. Additionally, the stability condition, proved in Section

3.3, states that the numerical solution approaches the finite-difference solution because at every time step the numerical error remains bounded. Both the stability and consistency imply that the numerical solution approaches the true solution as $n \to \infty$ and $k, h \to 0$; this is the concept of convergence.

4. NUMERICAL COMPUTATIONS

This section presents an exact solution test [3] which is used to examine the theoretical and computational accuracy of the finite-difference model. All computations were performed on the VAX 11/780 computer using double-precision complex arithmetic.

Consider the function

$$\eta(\xi, \tau) = e^{i\omega\tau} \sin(\omega\xi). \tag{41}$$

If we perform the partial derivations $\eta_{\xi\xi}$, $\eta_{\tau\tau}$, $\eta_{\xi\tau}$ and η_τ, and substitute them into the original dynamic motion equation (3a), we obtain an inhomogeneous equation of the form

$$\eta_{\tau\tau} + a(\xi)\eta_{\xi\xi} + b_{\eta\xi} + 2\beta_{\eta\xi\tau} + c\eta_\tau = F(\xi, \tau), \tag{42}$$

where

$$F(\xi, \tau) = [\{-\omega^2[a(\xi) + 1] + i\omega c\} \sin(\omega\xi) + (b\omega + 2\beta i\omega^2)\cos(\omega\xi)] e^{i\omega\tau}. \tag{43}$$

We regard $F(\xi, \tau)$ as simulating a forcing function and select input parameter values close to those of realistic physical parameters.

The associated initial and boundary conditions are taken as follows:

$$e^{i\omega\tau} \sin(\omega\xi)|_{\tau=0} = \sin(\omega\xi), \quad \text{initial condition;} \tag{44}$$

$$e^{i\omega\tau} \sin(\omega\xi)|_{\xi=0} = 0, \quad \text{fixed-end condition;} \tag{45}$$

$$e^{i\omega\tau} \sin(\omega\xi)|_{\xi=\xi_B} = e^{i\omega\tau} \sin(\omega\xi_B), \text{ such that } |e^{i\omega\tau}\sin(\omega\xi_B)| \leq 1 < \infty, \text{ free-end condition.} \tag{46}$$

We purposely select $\omega = i$ so that as $\tau \to \infty$, $\eta(\xi, \tau) \to 0$. In selecting a practical $\varepsilon = 5E + 4$, we find:

Problem	*Region*
Hyperbolic	$0 \leq \xi < 0.998$
Parabolic	$\xi = 0.998$
Elliptic	$0.998 < \xi \leq 1$

Note that the elliptic region is extremely small and considered negligible. This is usually true in many problems of interest.

The following inputs are selected for the computation:

$$\beta = 1/2, \quad c_T = 0.01, \quad c_N = 0.09, \quad \eta = 5E + 4, \quad \omega = 0.1i, \quad \Delta\tau = 1E - 5.$$

This problem is chosen such that the real part is zero and calculated up to 10,000 steps. The results are presented in Table 1, and show very good agreement with the exact solution.

If these computations had been carried out using the numerical ODE solution, they would have required calculation of a matrix exponential, a matrix inversion, two matrix–matrix multiplications and two matrix–vector multiplications. The total operation would be $O(N^3)$. The improved finite-difference scheme only requires the calculation of four scalar–vector multiplications, four vector–

Table 1 An exact solution test[a]

ξ \ τ	0.3052E − 04	0.3052E − 01	0.1114E + 00
0.1	0.1000E − 01 0.1000E − 01	0.9967E − 02 0.9969E − 02	0.9886E − 02 0.9889E − 02
0.2	0.2000E − 01 0.2000E − 01	0.1994E − 01 0.1994E − 01	0.1978E − 01 0.1977E − 01
0.3	0.3000E − 01 0.3000E − 01	0.2992E − 01 0.2991E − 01	0.2970E − 01 0.2967E − 01
0.4	0.4000E − 01 0.4001E − 01	0.3988E − 01 0.3988E − 01	0.3957E − 01 0.3956E − 01
0.5	0.5002E − 01 0.5002E − 01	0.4986E − 01 0.4986E − 01	0.4946E − 01 0.4946E − 01
0.6	0.6003E − 01 0.6003E − 01	0.5985E − 01 0.5985E − 01	0.5937E − 01 0.5937E − 01
0.7	0.7005E − 01 0.7005E − 01	0.6984E − 01 0.6984E − 01	0.6928E − 01 0.6928E − 01
0.8	0.8008E − 01 0.8008E − 01	0.7984E − 01 0.7984E − 01	0.7920E − 01 0.7919E − 01
0.9	0.9012E − 01 0.9012E − 01	0.8990E − 01 0.8984E − 01	0.8925E − 01 0.8912E − 01

[a]The table entries show both the exact (upper value) and the numerical (lower value) solution.

vector additions and a few scalar–scalar operations. The total operation is $O(N^2)$. It is evident that the improved finite-difference solution is much faster.

5. CONCLUSIONS

The previous solution [3] to a well-posed dynamic motion equation was made available using an efficient numerical ODE solution to a vibration problem of practical interest. This numerical solution improved the generality and computational efficiency of the previous special function solution. This efficient solution is now further improved by a stable explicit finite-difference scheme which increased the computation speed while maintaining the same accuracy. The theoretical development assures the stability, consistency and convergence of the solution. One test example was calculated and the answers compared very favorably with the exact solution, demonstrating the accuracy. This improved model, thus far, appears to give a more efficient numerical treatment of the transverse dynamics of a long thin cylinder in viscous axial fluid flow.

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